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APPROXIMATIONS TO THE EIGENVALUE RELATION FOR THE ORR–SOMMERFELD PROBLEM

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CONTENTS

A comprehensive study is made of the eigenvalue relation for the Orr–Sommerfeld problem. One of the major results obtained is a 'first approximation' to the eigenvalue relation which is uniformly valid along the entire curve of marginal stability. Two derivations of this approximation are given, one based on the use of uniform approximations to the solutions of the Orr–Sommerfeld equation and the other based on the differential equation satisfied by the eigenvalue relation itself. The theory is developed in detail for symmetrical flows in a channel but it is also applicable, with minor modifications, to flows of the boundary-layer type. Near the nose of the marginal curve the error associated with the approximation is of the order of ϵ^3, where $\epsilon = (\mathrm{i}\alpha R U_c')^{-\frac{1}{3}}$, and as $R \to \infty$ along the upper and lower branches of the marginal curve the errors are of the order of $\epsilon^{\frac{12}{5}}$ and $\epsilon^2 \ln \epsilon$ respectively. A comparison is also made with four heuristic approximations to the eigenvalue relation, two of which have been widely used in the past, and detailed calculations for plane Poiseuille flow clearly demonstrate the superiority of the uniform approximation.

1. Introduction

In the study of the stability of parallel shear flows, a great deal of effort has been devoted to the derivation of uniform asymptotic approximations to the solutions of the Orr–Sommerfeld equation, but relatively little attention has been given to the related problem of obtaining a uniform approximation to the eigenvalue relation. In this paper, therefore, we shall be concerned primarily with the derivation of a uniform 'first approximation' to the eigenvalue relation and with its relation to some of the heuristic approximations which have been used in the past.

We begin, in §2, by giving an exact formulation of the eigenvalue problem in terms of certain second-order Wronskians of the solutions of the Orr–Sommerfeld equation and later, in §6, it is shown that these Wronskians satisfy a pair of coupled third-order equations. Thus it is possible, at least in principle, to approximate the eigenvalue relation directly from the equations satisfied by these Wronskians. In the case of marginal stability, however, the eigenvalue relation can be substantially simplified. The largest error associated with these preliminary approximations is of order $(\alpha R)^{-2}$ for large values of the Reynolds number and it is with this approximation to the eigenvalue relation that we shall be concerned in the subsequent discussion.

In the usual method of approximating the eigenvalue relation it is necessary to first obtain approximations to the solutions. For this purpose one might consider using the composite approximations obtained by Eagles (1969) or Reid (1972) but two major difficulties immediately arise. First, as Reid (1972, 1974) has argued, these composite approximations are not uniformly valid in a full neighbourhood of the critical point and, second, they do not lead to the correct structure of the uniform approximation to the eigenvalue relation. Alternatively, however, we could consider the uniform approximations obtained by Lin (1957*a*, *b*, 1958) or Reid (1974). Lin's approximations are expressed in terms of the solutions of a certain fourth-order comparison equation whereas those obtained by Reid are expressed in terms of a certain class of generalized Airy functions. It might appear, therefore, that these two approaches lead to different results. In §3, however, it is shown that if uniform approximations to the solutions of the comparison equation are used in Lin's formulation then his results can be reduced to the simpler forms given by Reid. These approximations to the solutions of the Orr–Sommerfeld equation are then used in §5 to derive a 'first approximation' to the eigenvalue relation which is uniformly valid in a full neighbourhood of the critical point. An alternate derivation of this approximation, based on a simple matching technique, is given in §6.

For computational purposes it is helpful to rewrite the approximations in terms of slowly varying functions only. This can be done, as shown in §7, in terms of certain Tietjens-type functions and the results obtained for plane Poiseuille flow are found to be in very good agreement with the 'exact' numerical results of Reynolds & Potter (1967) and Orszag (1971). Finally, in §8 we give a short discussion of four heuristic approximations to the eigenvalue relation in an endeavour to relate the present uniform approximation to two of the simpler but non-uniform approximations which have been widely used in the past.

2. Formulation of the eigenvalue problem

Consider then the Orr–Sommerfeld equation which can be written in the form

$$(\mathrm{i}\alpha R)^{-1}(\mathrm{D}^2-\alpha^2)^2\phi-\{(U-c)(\mathrm{D}^2-\alpha^2)\phi-U''\phi\}=0, \tag{2.1}$$

where $\phi(z)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}$ is the stream function of the disturbance in the usual normal mode analysis, $U(z)$ is the basic velocity distribution, R is the Reynolds number, and $\mathrm{D}=\mathrm{d}/\mathrm{d}z$. A point z_c at which $U-c=0$ is often called a critical point. If, as we shall suppose, $U'_c\equiv U'(z_c)\neq 0$ then z_c can also be described as a simple turning point of equation (2.1).

The precise form of the eigenvalue relation depends, of course, on the class of basic flows considered and, for simplicity, we shall consider only symmetrical flows in a channel. With minor modifications, however, the present results can also be applied to flows of the boundary-layer type. Because of the symmetry of the basic flow we can treat the even and odd solutions separately. It is known that the odd solutions are stable and we will therefore consider only the even solutions which must then satisfy the boundary conditions

$$\phi=\mathrm{D}\phi=0 \quad \text{at} \quad z=z_1 \quad \text{and} \quad \mathrm{D}\phi=\mathrm{D}^3\phi=0 \quad \text{at} \quad z=z_2. \tag{2.2}$$

We will suppose further that $U'(z)>0$ on the interval $[z_1,z_2)$ and, without loss of generality, that $U(z_1)=0$ and $U(z_2)=1$. An important flow of this type is plane Poiseuille flow for which $U(z)=1-z^2$ with $z_1=-1$ and $z_2=0$, and this flow will be used for illustrative purposes throughout the paper.

In discussing asymptotic approximations to the solutions of equation (2.1) and to the eigenvalue relation it is convenient to introduce the parameter

$$\epsilon=(\mathrm{i}\alpha RU'_c)^{-\frac{1}{3}}. \tag{2.3}$$

We shall suppose, of course, that $0<|\epsilon|\ll 1$ and, in the case of marginal stability, that $\mathrm{ph}\,\epsilon=-\frac{1}{6}\pi$.

Now let $\phi_i(z)$ $(i=1,2,3,4)$ denote a linearly independent set of solutions of equation (2.1). The general solution of the equation can then be written in the form

$$\phi=C_1\phi_1+C_2\phi_2+C_3\phi_3+C_4\phi_4. \tag{2.4}$$

On imposing the boundary conditions (2.2), we obtain a system of four linear homogeneous equations for the constants C_i. For a non-trivial solution, the determinant of the coefficients must vanish and this leads to the eigenvalue relation

$$F(\alpha,c,\epsilon)=\begin{vmatrix}\phi_{11} & \phi_{21} & \phi_{31} & \phi_{41}\\ \phi'_{11} & \phi'_{21} & \phi'_{31} & \phi'_{41}\\ \phi'_{12} & \phi'_{22} & \phi'_{32} & \phi'_{42}\\ \phi'''_{12} & \phi'''_{22} & \phi'''_{32} & \phi'''_{42}\end{vmatrix}=0, \tag{2.5}$$

where $\phi_{i\nu}=\phi_i(z_\nu)$ $(i=1,2,3,4;\ \nu=1,2)$. If we now let $\mathscr{W}(u,v)(z)$ denote the usual second order Wronskian of $u(z)$ and $v(z)$ then, by Laplace's expansion of a determinant by complementary minors, we obtain

$$\begin{aligned}F(\alpha,c,\epsilon)=&\ \mathscr{W}(\phi_2,\phi_3)(z_1)\,\mathrm{d}\mathscr{W}(\phi'_1,\phi'_4)(z_2)/\mathrm{d}z\\ &-\mathscr{W}(\phi_1,\phi_3)(z_1)\,\mathrm{d}\mathscr{W}(\phi'_2,\phi'_4)(z_2)/\mathrm{d}z\\ &+\mathscr{W}(\phi_1,\phi_2)(z_1)\,\mathrm{d}\mathscr{W}(\phi'_3,\phi'_4)(z_2)/\mathrm{d}z\\ &+\mathscr{W}(\phi_3,\phi_4)(z_1)\,\mathrm{d}\mathscr{W}(\phi'_1,\phi'_2)(z_2)/\mathrm{d}z\\ &-\mathscr{W}(\phi_2,\phi_4)(z_1)\,\mathrm{d}\mathscr{W}(\phi'_1,\phi'_3)(z_2)/\mathrm{d}z\\ &+\mathscr{W}(\phi_1,\phi_4)(z_1)\,\mathrm{d}\mathscr{W}(\phi'_2,\phi'_3)(z_2)/\mathrm{d}z=0.\end{aligned} \tag{2.6}$$

The importance of being able to express the eigenvalue relation in terms of these second-order Wronskians is due to the fact that if u and v denote *any* two solutions of the Orr–Sommerfeld equation then it is possible, as will be shown in § 6, to derive a pair of coupled third-order equations for $\mathscr{W}(u,v)(z)$ and $\mathscr{W}(u',v')(z)$. From these equations it would then be possible, at least in principle, to derive approximations to the Wronskians and hence to the eigenvalue relation without having to obtain approximations to the solutions themselves.

Consider next the possibility of simplifying the eigenvalue relation (2.6). For this purpose it is necessary to characterize the behaviour of the solutions as $\epsilon \to 0$ and, in the terminology of Wasow (1969), we require therefore that ϕ_1 be well balanced, that ϕ_2 be balanced at z_1 and z_2, and that ϕ_3 and ϕ_4 be recessive at z_2 and z_1 respectively. Although we do not, in general, have exact representations for the solutions, these conditions suffice to define them uniquely (apart from arbitrary mutiplicative factors and modulo an arbitrary additive multiple of ϕ_1 in the case of ϕ_2). The existence of solutions having these asymptotic properties has been proved by Lin & Rabenstein (1969) for the class of basic flows considered here but we will make no essential use of this fact.

In discussing the approximations to equation (2.6) it is necessary to consider the outer expansions of the solutions. These are well-known from the heuristic theory and need only be summarized briefly here. Thus, we have

$$\phi_1(z) \sim \phi_1^{(0)}(z) \quad \text{and} \quad \phi_2(z) \sim \phi_2^{(0)}(z), \tag{2.7}$$

where $\phi_1^{(0)}$ and $\phi_2^{(0)}$ are solutions of Rayleigh's equation, i.e. the reduced form of equation (2.1). These 'inviscid' solutions can be expressed in the form

$$\phi_1^{(0)}(z) = (z-z_c)\, P_1(z) \tag{2.8}$$

and

$$\phi_2^{(0)}(z) = P_2(z) + (U_c''/U_c')\, \phi_1^{(0)}(z) \ln(z-z_c), \tag{2.9}$$

where $P_1(z)$ and $P_2(z)$ are analytic at z_c with $P_1(z_c) = P_2(z_c) = 1$ and, to fix the normalization, $P_2'(z_c) = 0$. This approximation to ϕ_1 is valid in a full (complex) neighbourhood of z_c and the error associated with it is of order ϵ^3. The approximation to ϕ_2, however, is valid only if $|z-z_c| \gg |\epsilon|$ and the phase of $z-z_c$ is suitably restricted. In particular, when z and z_c are real, if we take $\mathrm{ph}\,(z-z_c) = 0$ for $z-z_c > 0$ then, following Tollmien (1929), we must take $\mathrm{ph}\,(z-z_c) = -\pi$. We can therefore use $\phi_2^{(0)}$ to approximate ϕ_2 at z_1 and z_2 with an error which is again of order ϵ^3 provided $z_c - z_1 \gg |\epsilon|$ and $z_2 - z_c \gg |\epsilon|$. The outer expansions of ϕ_3 and ϕ_4 are of W.K.B.J. type and they can be written in the form

$$\phi_3(z) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}} \left(\frac{U-c}{U_c'}\right)^{-\frac{5}{4}} \exp\{-\epsilon^{-\frac{3}{2}}Q(z)\}\{1-\epsilon^{\frac{3}{2}}G_1(z)+O(\epsilon^3)\} \tag{2.10}$$

and

$$\phi_4(z) = \mathrm{i}\tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}} \left(\frac{U-c}{U_c'}\right)^{-\frac{5}{4}} \exp\{+\epsilon^{-\frac{3}{2}}Q(z)\}\{1+\epsilon^{\frac{3}{2}}G_1(z)+O(\epsilon^3)\}, \tag{2.11}$$

where

$$Q(z) = \int_{z_c}^{z} \left(\frac{U-c}{U_c'}\right)^{\frac{1}{2}} \mathrm{d}z \tag{2.12}$$

and

$$G_1(z) = \left(\frac{101}{48}\frac{U'}{U-c} + \frac{23}{24}\frac{U''}{U'}\right)\left(\frac{U_c'}{U-c}\right)^{\frac{1}{2}} - \int_{z_c}^{z} \left\{\frac{23}{24}\left(\frac{U'''}{U'} - \frac{U''^2}{U'^2}\right) - \tfrac{1}{2}\alpha^2\right\}\left(\frac{U_c'}{U-c}\right)^{\frac{1}{2}} \mathrm{d}z. \tag{2.13}$$

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These approximations are again valid only if $|z-z_c| \gg |\epsilon|$ and the phase of $z-z_c$ is suitably restricted as discussed, for example, by Lakin & Reid (1970). The terms involving $G_1(z)$ are not needed for the present discussion but they will be required later in § 6.

The first simplification of equation (2.6) we wish to consider is based on the observation, which followed from equations (2.10) and (2.11), that ϕ_3 and ϕ_4 are necessarily dominant at z_1 and z_2 respectively, and hence the first two terms in equation (2.6) are also dominant, the second two terms are balanced, and the last two terms are recessive. Thus, with an exponentially small error, we can approximate the eigenvalue relation by

$$G(\alpha, c, \epsilon) = \mathscr{W}(\phi_2, \phi_3)(z_1)\, \mathrm{d}\mathscr{W}(\phi_1', \phi_4')(z_2)/\mathrm{d}z - \mathscr{W}(\phi_1, \phi_3)(z_1)\, \mathrm{d}\mathscr{W}(\phi_2', \phi_4')(z_2)/\mathrm{d}z = 0. \tag{2.14}$$

This approximation is equivalent, as Lin (1955, pp. 35–36) has shown, to neglecting ϕ_{32}' and ϕ_{32}'''' compared to ϕ_{31} and ϕ_{31}' in equation (2.5) and similarly neglecting ϕ_{41} and ϕ_{41}' compared to ϕ_{42}' and ϕ_{42}''''. To obtain a more precise estimate of the error associated with this approximation we first note that the largest error arises from the neglect of ϕ_{32}'''' compared to ϕ_{31}'. Thus, consider the ratio ϕ_{32}''''/ϕ_{31}' which can easily be estimated by using the W.K.B.J. approximation (2.10) to ϕ_3. If $0 < c < 1$, with c bounded away from both 0 and 1, then a simple calculation gives

$$|\phi_{32}''''/\phi_{31}'| \sim (1-c)^{\frac{1}{4}} c^{\frac{3}{4}} \alpha R \exp\{-(\tfrac{1}{2}\alpha R)^{\frac{1}{2}} \lambda(c)\}, \tag{2.15}$$

where

$$\lambda(c) = \int_{z_1}^{z_2} |U-c|^{\frac{1}{2}}\, \mathrm{d}z. \tag{2.16}$$

To obtain a numerical estimate of (2.15) consider plane Poiseuille flow for which

$$\lambda(c) = \tfrac{1}{2}\sqrt{c} - \tfrac{1}{2}(1-c)\, \mathrm{artanh}\sqrt{c} + \tfrac{1}{4}\pi(1-c). \tag{2.17}$$

For values of α, c, and R near the nose of the marginal curve equation (2.15) then gives $|\phi_{32}''''/\phi_{31}'| \approx 3 \times 10^{-12}$ which is many orders of magnitude smaller than any of the terms retained in the subsequent analysis.

On the lower branch of the curve of marginal stability α and c both tend to zero as $R \to \infty$ and in this case it is necessary to approximate ϕ_{31}' by the first term of its inner expansion. A short calculation then gives

$$|\phi_{32}''''/\phi_{31}'| \sim \text{constant}\, R^{\frac{9}{14}} \exp(-\text{constant}\, R^{\frac{3}{7}}) \tag{2.18}$$

as $R \to \infty$ along the lower branch of the curve of marginal stability. The constants appearing in this equation are both positive and independent of R. They can be evaluated in terms of $U'(z_1)$, $U''(z_1)$, and $\int_{z_1}^{z_2} U^2\, \mathrm{d}z$ but the results are somewhat complicated and are not needed for the present purposes. For flows without inflection points, α and c also tend to zero as $R \to \infty$ on the upper branch of the curve of marginal stability. In this case, however, equation (2.15) remains valid and leads to the estimate

$$|\phi_{32}''''/\phi_{31}'| \sim \text{constant}\, R^{\frac{17}{22}} \exp(-\text{constant}\, R^{\frac{5}{11}}) \tag{2.19}$$

as $R \to \infty$ along the upper branch of the curve of marginal stability.

Thus, when $0 \leqslant c < 1$, the error associated with the approximation (2.14) is exponentially small as $R \to \infty$ and it is very small numerically for values of R near the nose of the marginal curve. When $c \uparrow 1$, however, we have a situation where two simple turning points coalesce to form a single turning point of second order at $z = z_2$. This situation lies outside the scope of the present theory and does not occur anywhere on the curve of marginal stability.

In discussing the next approximation it is convenient to rewrite equation (2.14) in a somewhat different form. For this purpose let

$$\Phi = A\phi_1 + \phi_2 \quad \text{and} \quad \hat{\Phi} = B\phi_1 + \phi_2, \tag{2.20}$$

where
$$\left.\begin{aligned} A(\alpha, c, \epsilon) &= -\phi'_{22}/\phi'_{12} \\ B(\alpha, c, \epsilon) &= -\phi'''_{22}/\phi'''_{12}. \end{aligned}\right\} \tag{2.21}$$
and

Equation (2.14) can be written in the equivalent form

$$H(\alpha, c, \epsilon) = \mathscr{W}(\Phi, \phi_3)(z_1) - \frac{\phi'''_{12}\,\phi'_{42}}{\phi'_{12}\,\phi'''_{42}} \mathscr{W}(\hat{\Phi}, \phi_3)(z_1) = 0. \tag{2.22}$$

We will now assume that z_c is closer to z_1 than to z_2, as it happens to be in most problems. This assumption then permits the neglect of those terms in the asymptotic expansions of ϕ_2 and ϕ_4 near z_2 which are recessive on a length scale $z_2 - z_c$. The distinction here is between expansions which are valid in the usual Poincaré sense as opposed to being valid in the complete sense of Watson (Olver 1974, p. 543). The outer expansions (2.7) and (2.11) then show that

$$\left.\begin{aligned} \frac{\phi'''_{12}}{\phi'_{12}} &= \alpha^2 + \frac{U''(z_2)}{1-c} + O(\epsilon^3) \\ \text{and} \qquad \frac{\phi'_{42}}{\phi'''_{42}} &= \frac{U'_c}{1-c}\,\epsilon^3 + O(\epsilon^{\frac{9}{2}}). \end{aligned}\right\} \tag{2.23}$$

We also see that the quantities A and B defined by equations (2.21) must have expansions of the form

$$A(\alpha, c, \epsilon) = \sum_{s=0}^{\infty} \epsilon^{3s} A^{(s)}(\alpha, c) \quad \text{and} \quad B(\alpha, c, \epsilon) = \sum_{s=0}^{\infty} \epsilon^{3s} B^{(s)}(\alpha, c), \tag{2.24}$$

where
$$A^{(0)}(\alpha, c) = B^{(0)}(\alpha, c) = -\phi^{(0)\prime}_{22}/\phi^{(0)\prime}_{12}. \tag{2.25}$$

This last result shows that $\mathscr{W}(\hat{\Phi}, \phi_3)(z_1) = \mathscr{W}(\Phi, \phi_3)(z_1) + O(\epsilon^3)$ and hence, with a relative error of the order of ϵ^6, we can approximate the eigenvalue relation by

$$\Delta(\alpha, c, \epsilon) = \mathscr{W}(\Phi, \epsilon\phi_3)(z_1) = 0, \tag{2.26}$$

where the factor ϵ has been introduced for scaling purposes. This form of the eigenvalue relation is exact for flows of the boundary-layer type provided A is determined from the condition that Φ is bounded as $z \to \infty$.

In the subsequent discussion of the various approximations to equation (2.26) it is convenient to let

$$\Delta(z) \equiv \Delta(\alpha, c, \epsilon; z) = \mathscr{W}(\Phi, \epsilon\phi_3)(z) \tag{2.27}$$

so that the eigenvalue relation in this approximation becomes simply $\Delta(z_1) = 0$. Our problem then is to derive a 'first approximation' to $\Delta(z)$ which is uniformly valid in a domain of the (complex) z-plane containing z_1.

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3. Uniform approximations to the solutions

In the derivation of uniform asymptotic approximations to the solutions of equation (2.1), it is customary to make a preliminary transformation of the form

$$\phi(z) = \{\eta'(z)\}^m \chi(\eta), \tag{3.1}$$

where the Langer variable $\eta(z)$ is defined by

$$\eta(z) = \left[\frac{3}{2}\int_{z_c}^{z}\left(\frac{U-c}{U_c}\right)^{\frac{1}{2}} \mathrm{d}z\right]^{\frac{2}{3}}. \tag{3.2}$$

The exponent m in equation (3.1) is often chosen so that the transformed equation is in normal form, i.e. so that it does not contain χ''', and this condition requires that $m = -\frac{3}{2}$. Clearly the final results must be independent of m and, when dealing with equations which are not given initially in normal form, a more convenient choice would appear to be $m = 0$. With that choice for m we then find that equation (2.1) becomes

$$\epsilon^3(\chi^{\mathrm{iv}} + f_0\chi''') - (\eta + \epsilon^3 f_1)\chi'' - (g_0 + \epsilon^3 g_1)\chi' - (h_0 + \epsilon^3 h_1)\chi = 0, \tag{3.3}$$

where

$$\left.\begin{aligned} f_0(\eta) &= 6\gamma, \\ f_1(\eta) &= -(4\gamma' + 11\gamma^2 - 2\alpha^2\eta'^{-2}), \\ g_0(\eta) &= \eta\gamma, \\ g_1(\eta) &= -(\gamma'' + 7\gamma'\gamma + 6\gamma^3 - 2\alpha^2\gamma\eta'^{-2}), \\ h_0(\eta) &= -(2\eta\gamma' + 6\eta\gamma^2 + 5\gamma + \alpha^2\eta\eta'^{-2}), \\ h_1(\eta) &= -\alpha^4\eta'^{-4}, \end{aligned}\right\} \tag{3.4}$$

and

$$\gamma(\eta) = \eta''/\eta'^2. \tag{3.5}$$

It would, of course, have been possible to develop the present theory in a more general form by assuming only that the coefficients $f_0, f_1, \ldots, h_1$ which appear in equation (3.3) are analytic functions of η in some neighbourhood of $\eta = 0$. The general structure of the theory then depends in a crucial way on the value of $g_0(0)$ and to a lesser extent on the value of $h_0(0)$. Since we are primarily concerned with the eigenvalue relation for the Orr–Sommerfeld equation we shall suppose, for simplicity, that $f_0, f_1, \ldots, h_1$ have the forms given by equations (3.4) and it is then easily seen that

$$g_0(0) = 0 \quad \text{and} \quad h_0(0) = -5\gamma(0) = -U_c''/U_c'. \tag{3.6}$$

In discussing the solutions of equation (3.3) it is convenient to consider seven solutions which will be denoted by $U_0(\eta)$, $U_k(\eta)$, and $V_k(\eta)$ $(k = 1, 2, 3)$. As in the case of the solutions of the untransformed equation, these solutions of the transformed equation can be uniquely defined (again to within multiplicative factors and modulo an arbitrary additive multiple of U_0 in the case of the U_k) in terms of their asymptotic properties. Thus, we require that U_0 be well balanced, that U_k be (purely) balanced in $\mathbf{T}_k$, and that V_k be recessive in $\mathbf{S}_k$, where $\mathbf{T}_k$ and $\mathbf{S}_k$ are the sectors shown in figure 1. It then follows immediately that we must have

$$\left.\begin{aligned} \phi_1(z) &\equiv C_1 U_0(\eta), \quad \phi_2(z) \equiv C_0 U_0(\eta) + C_2 U_3(\eta), \\ \phi_3(z) &\equiv C_3 V_1(\eta), \quad \text{and} \quad \phi_4(z) \equiv C_4 V_2(\eta), \end{aligned}\right\} \tag{3.7}$$

where the values of the constants $C_0, C_1, \ldots, C_4$ depend only on the way in which the solutions have been normalized. Without loss of generality, however, we can fix the normalization of the solutions so that

$$C_0 = 0 \quad \text{and} \quad C_1 = C_2 = C_3 = C_4 = 1, \tag{3.8}$$

and we will suppose that this has been done. These seven solutions of equation (3.3) cannot, of course, be linearly independent but must be related by three connection formulae. Approximations to these connection formulae can easily be obtained directly from the uniform approximations to the solutions given below but they are not needed for the present purposes.

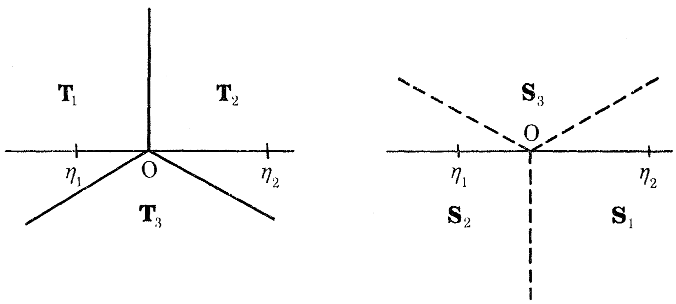

FIGURE 1. The Stokes lines (left) and the anti-Stokes lines (right) in the η-plane for U_c' real and positive.

Uniform approximations to the solutions of equation (3.3) and hence to the solutions of equation (2.1) have been derived by Lin (1957*a*, *b*, 1958) and Reid (1974) by methods which would appear to be quite different and it is of some interest, therefore, to consider briefly the relation between these two approaches. According to Lin's theory, asymptotic solutions of equation (3.3) can be found in the form†

$$\chi = Au + Bu' + \epsilon^3(Cu'' + Du'''), \tag{3.9}$$

where u is a solution of the comparison equation

$$\epsilon^3 u^{iv} - (\eta u'' + \alpha u' + \beta u) = 0. \tag{3.10}$$

In equation (3.9), the slowly-varying coefficients A, B, C and D all have expansions of the form

$$A \equiv A(\eta, \epsilon) = \sum_{s=0}^{\infty} A_s(\eta)\, \epsilon^{3s}. \tag{3.11}$$

Similarly, in equation (3.10), α and β must also have expansions of the form

$$\alpha \equiv \alpha(\epsilon) = \sum_{s=0}^{\infty} \alpha_s\, \epsilon^{3s}. \tag{3.12}$$

The coefficients A_s, B_s, C_s and D_s can be determined by solving the differential equations which they satisfy. In doing so, however, the constants α_s and β_s must be chosen so that the solutions are all analytic at $\eta = 0$. Although this can be done in principle, it is extremely complicated in practice. Alternatively, as Lin (1957*b*) has shown, these difficulties can be avoided by the use of a simple 'matching' technique. [The matching here is not of the term-by-term type used in the usual method of matched asymptotic expansions.] In this latter approach, outer expansions to the solutions of the comparison equation are substituted into the expansion (3.9) and the results

† The coefficients A and B which appear in this equation are not related to the quantities defined by equation (2.21) and the α which appears in equation (3.10) is not related to the wavenumber.

are then matched to the corresponding outer expansions to the solutions of equation (3.3). In this way Lin (1958) was able to determine A_0, B_0, C_0 and D_0; he also found that

$$\alpha_0 = g_0(0) = 0 \quad \text{and} \quad \beta_0 = h_0(0) = -U_c''/U_c'.$$

Although this method has been formally successful, nevertheless there are two remaining difficulties which have prevented it from being used in actual calculations. One of these difficulties is concerned with the truncation of the expansion (3.9) and the other with the approximation of the solutions of the comparison equation (3.10). We will return to these questions after describing the method developed by Reid (1974).

According to this theory, uniform asymptotic approximations to the solutions of equation (3.3) of balanced and dominant-recessive type can be expressed to all orders in terms of the generalized Airy functions

$$A_k(\zeta, p, q) = \frac{1}{2\pi i}\int_{L_k} t^{-p} (\ln t)^q \exp(\zeta t - \tfrac{1}{3}t^3)\,dt \tag{3.13}$$

and

$$B_k(\zeta, p, q) = \frac{1}{2\pi i}\int_{\infty\exp[\frac{2}{3}(k-1)\pi i]}^{(0+)} t^{-p} (\ln t)^q \exp(\zeta t - \tfrac{1}{3}t^3)\,dt, \tag{3.14}$$

where $p = 0, \pm 1, \pm 2, \ldots$, $q = 0, 1, 2, \ldots$, L_k are the usual Airy contours, and a branch cut has been placed along the positive real axis in the t-plane so that $0 \leqslant \text{ph}\, t < 2\pi$. For simplicity we shall also let $A_k(\zeta, p, 0) \equiv A_k(\zeta, p)$. This class of generalized Airy functions was first introduced by Reid (1972) where further details are given. The concept of a 'first approximation' also plays an important rôle in this theory and is related to the truncation problem mentioned above. A further distinction must be made between first approximations to the solutions of equation (3.3) and a first approximation to the eigenvalue relation (2.27).

Consider then the first approximations

$$U_k(\eta) \sim \mathscr{G}(\eta) - \epsilon\{\mathscr{A}(\eta)\, B_k(\zeta, 2, 1) + \epsilon^2\mathscr{B}(\eta)\, B_k(\zeta, 1, 1) + \epsilon\mathscr{C}(\eta)\, B_k(\zeta, 0, 1)\} \tag{3.15}$$

and

$$V_k(\eta) \sim a(\eta)\, A_k(\zeta, 2) + \epsilon^2 b(\eta)\, A_k(\zeta, 1) + \epsilon c(\eta)\, A_k(\zeta, 0). \tag{3.16}$$

The slowly varying coefficients in these approximations can all be expressed in terms of three quantities associated with the outer expansions of U_0, U_3, and V_1. Thus, as in §2, we have

$$U_0(\eta) \sim U_0^{(0)}(\eta) \quad \text{and} \quad U_3(\eta) \sim U^{(0)}(\eta), \tag{3.17}$$

where $U_0^{(0)}$ and $U^{(0)}$ are solutions of the reduced form of equation (3.3). They can also be written in the form

$$U_0^{(0)}(\eta) = \eta Q_1(\eta) \tag{3.18}$$

and

$$U^{(0)}(\eta) = Q_2(\eta) + (U_c''/U_c')\, U_0^{(0)}(\eta) \ln \eta, \tag{3.19}$$

where $Q_1(\eta)$ and $Q_2(\eta)$ are analytic at $\eta = 0$ with $Q_1(0) = Q_2(0) = 1$ and, to fix the normalization, $Q_2'(0) = 0$. This approximation to U_0 is actually uniformly valid in a full neighbourhood of $\eta = 0$ with an error of the order of ϵ^3. The approximation to U_3 is valid in the sense of Watson only for finite values of η in $\mathbf{T}_3$ and the error is then also of the order of ϵ^3; it remains valid, however, in the sense of Poincaré in the larger sector $\mathbf{S}_1 \cup \mathbf{S}_2$. The outer expansion of V_1 is of W.K.B.J. type and is given by

$$V_1(\eta) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{5}{4}}(\eta\eta'^2)^{-\frac{5}{4}} \exp(-\tfrac{2}{3}\epsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}})\,\{1 - \epsilon^{\frac{3}{2}}H_1(\eta) + O(\epsilon^3)\} \tag{3.20}$$

where

$$H_1(\eta) = \tfrac{101}{48}\eta^{-\frac{3}{2}} + 9\gamma\eta^{-\frac{1}{2}} - \int_0^{\eta} (\tfrac{23}{4}\gamma' + \tfrac{23}{8}\gamma^2 - \tfrac{1}{2}\alpha^2\eta'^{-2})\,\eta^{-\frac{1}{2}}\,d\eta. \tag{3.21}$$

This approximation to V_1 is valid in the sense of Watson for finite values of η in $\mathbf{T}_2 \cup \mathbf{T}_3$ and the relative error is then of the order of ϵ^3; it also remains valid in the sense of Poincaré in the sector $-\frac{7}{6}\pi < \mathrm{ph}\,\eta < \frac{5}{6}\pi$. In terms of these quantities we have

$$\left.\begin{aligned} a(\eta) &= \eta^{-1}U_0^{(0)}(\eta), \quad a(\eta)+\eta c(\eta) = \eta'^{-\frac{5}{2}}, \\ b(\eta) &= 2\eta^{-1}c(\eta)+\eta'^{-\frac{5}{2}}\eta^{-\frac{1}{2}}\{H_1(\eta)-\tfrac{101}{48}\eta^{-\frac{3}{2}}\}, \\ \mathscr{G}(\eta) &= Q_2(\eta)+(U_c''/U_c')\,[\ln\epsilon+\psi(2)+2\pi\mathrm{i}]\,U_0^{(0)}(\eta), \\ \text{and}\qquad \mathscr{A}(\eta)/a(\eta) &= \mathscr{B}(\eta)/b(\eta) = \mathscr{C}(\eta)/c(\eta) = U_c''/U_c'. \end{aligned}\right\} \tag{3.22}$$

For numerical purposes it is more convenient to compute the inviscid solutions in terms of z rather than η. This can easily be done since

$$U_0^{(0)}(\eta) \equiv \phi_1^{(0)}(z), \quad U^{(0)}(\eta) \equiv \phi_2^{(0)}(z). \tag{3.23}$$

and

$$Q_2(\eta) = P_2(z)-(U_c''/U_c')\,\phi_1^{(0)}(z)\ln[\eta/(z-z_c)]. \tag{3.24}$$

We also note that $H_1(\eta) \equiv G_1(z)$. Thus, the slowly varying coefficients in the approximations (3.15) and (3.16) can all be expressed ultimately in terms of the regular inviscid solution $\phi_1^{(0)}(z)$, the regular part of the singular inviscid solution $\phi_2^{(0)}(z)$, and a regularized form of the coefficient $G_1(z)$ which appears in the outer expansions of dominant-recessive type.

Consider now the relationship between these two methods. Rabenstein (1958) has shown that the solutions of the comparison equation (3.10) can be expressed in terms of Laplace integrals but they are too complicated to be used directly in the expansion (3.9). From the integral representation of the solutions he then derived inner and outer expansions which were, of course, automatically matched. Neither expansion, however, is uniformly valid and hence neither expansion is suitable for use in equation (3.9). What are needed here are uniform approximations to the solutions of the comparison equation and they provide the necessary link between the two methods. To illustrate the essential points involved here consider the solution $v_1(\eta)$ of equation (3.10) which is recessive in $\mathbf{S}_1$. It is immediately evident that this solution must have a uniform first approximation of the form

$$v_1(\eta) = a(\eta)\,A_1(\zeta,2)+\epsilon^2 b(\eta)\,A_1(\zeta,1)+\epsilon c(\eta)\,A_1(\zeta,0), \tag{3.25}$$

and a simple calculation shows that the slowly varying coefficients in this approximation are given by

$$\left.\begin{aligned} a(\eta) &= (\beta_0\eta)^{-\frac{1}{2}}J_1(2\beta_0^{\frac{1}{2}}\eta^{\frac{1}{2}}), \quad a(\eta)+\eta c(\eta) = 1, \\ \text{and}\qquad b(\eta) &= \eta^{-1}[2c(\eta)-\beta_0]. \end{aligned}\right\} \tag{3.26}$$

If this approximation to $v_1(\eta)$ and the corresponding approximations to its derivatives are substituted into the expansion (3.9) then, in a first approximation to $V_1(\eta)$, it is found that we must know not only A_0, B_0, C_0 and D_0 but also the quantity $B_1+\eta D_1$. This is not altogether surprising since it is known that D_0 appears only in the combination $B_0+\eta D_0$ and that $B_0+\eta D_0 = 0$. Thus, in effect, it is still necessary to determine only four coefficients in the expansion (3.9). A determination of the quantity $B_1+\eta D_1$ has been given previously by Lakin & Reid (1970)† who showed that it could be expressed in terms of A_0, B_0, C_0 and $G_1(z)$. This again shows the importance of the term $G_1(z)$ which appears in the outer expansions of W.K.B.J. type and which was not considered in any of the older heuristic theories.

† In that work it was not fully recognized that, to this order of approximation, B_1 and D_1 appear only in the combination $B_1+\eta D_1$. As a result B_1 should be replaced by $B_1+\eta D_1$ in equations (6.8), (6.9), (6.13) and (6.14). These corrections, however, do not affect any of the results obtained in that paper.

Thus, by using the results given by Lakin & Reid (1970) for the relevant coefficients in the expansion (3.9), we can pass directly from the uniform approximation to $v_1(\eta)$ to the corresponding uniform approximation to $V_1(\eta)$. This can also be done for the solutions of balanced type. The calculations involved in this approach are rather lengthy and for that reason they have not been given in detail. From this discussion, however, we can conclude that the theories of Lin and Reid lead to uniform 'first approximations' to the solutions of the Orr–Sommerfeld equation which are identical. A similar conclusion would be expected to hold for higher approximations though it is doubtful that they would ever be needed.

4. Error estimates for the uniform approximations

In the derivation of the first approximations to the solutions of equation (3.3) it was assumed that α and c were bounded away from zero. As $R \to \infty$ along the upper and lower branches of the curve of marginal stability, however, they both tend to zero and in these limits $\eta(z_1)$ is of order $\epsilon^{\frac{3}{5}}$ and ϵ respectively. Ideally, of course, we would like to have error bounds for the approximations which are valid in a full neighbourhood of the turning point but that would require the development of a theory of error bounds similar to the one which Olver (1974) has developed for second-order equations with a simple turning point. We can, however, obtain simple error estimates for the inner, intermediate and outer expansions of the solutions which correspond to η being of order ϵ, $\epsilon^{\frac{3}{5}}$ and 1 respectively, and we then have the orders of magnitude shown in table 1.

Table 1. Orders of magnitude for symmetrical flows in a channel without inflection points

	η	ζ	α	c	$A^{(0)}$	$\eta A^{(0)}$
inner	ϵ	1	$\epsilon^{\frac{1}{2}}$	ϵ	ϵ^{-1}	1
intermediate	$\epsilon^{\frac{3}{5}}$	$\epsilon^{-\frac{2}{5}}$	$\epsilon^{\frac{3}{10}}$	$\epsilon^{\frac{3}{5}}$	$\epsilon^{-\frac{3}{5}}$	1
outer	1	ϵ^{-1}	1	1	1	1

Consider first the solution of well balanced type. It has a uniform expansion of the form

$$U_0(\eta) = \sum_{s=0}^{\infty} \epsilon^{3s} U_0^{(s)}(\eta), \tag{4.1}$$

where $U_0^{(0)}(\eta) = O(\eta)$ as $\eta \to 0$ and $U_0^{(1)}(0) = \frac{2}{3}\alpha^2 - (U_c^{\text{iv}}/U_c'')$. Thus, if $U_c^{\text{iv}} \neq 0$ then $U_0^{(0)}(\eta)$ provides an approximation to the inner, intermediate and outer expansions of $U_0(\eta)$ with a *relative* error of order ϵ^2, $\epsilon^{\frac{12}{5}}$ and ϵ^3 respectively, and we shall denote such error estimates by $O(\epsilon^2, \epsilon^{\frac{12}{5}}, \epsilon^3)$. If $U_c^{\text{iv}} = 0$, as it does for plane Poiseuille flow, then the error is of order ϵ^3 for all η.

Consider next $V_1(\eta)$ and suppose, without loss of generality, that its outer expansion in $\mathbf{T}_2 \cup \mathbf{T}_3$ does not contain powers of $\ln \epsilon$. Its inner expansion, however, must necessarily contain powers of $\ln \epsilon$. From a consideration of the four terms in the inner expansion of $V_1(\eta)$ given by Reid (1972) it then follows that the relative errors associated with the approximation (3.16) are $O(\epsilon^3 \ln \epsilon, \epsilon^3, \epsilon^3)$, where η must lie in the sector $\mathbf{T}_2 \cup \mathbf{T}_3$ for the intermediate and outer estimates.

In deriving a consistent first approximation to the eigenvalue relation it is found that the term involving $\mathscr{b}(\eta)$ in equation (3.16) must be omitted. This is not unexpected, however, since Lakin & Reid (1970) had observed that when outer expansions are used to approximate the eigenvalue relation then it is necessary to retain two terms in the outer expansion of ϕ_3' but only one term in the

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outer expansion of ϕ_3. It is also found that the final form of the approximation can be greatly simplified if it is expressed in terms of generalized Airy functions with $p = 0, \pm 1$ rather than $p = 0, 1, 2$. Thus, on omitting the term involving $\mathscr{b}(\eta)$ in equation (3.16) and then using the recursion formula

$$A_k(\zeta, 2) = \zeta A_k(\zeta, 1) - A_k(\zeta, -1) \tag{4.2}$$

we obtain

$$\begin{aligned} \epsilon V_k(\eta) &\sim \mathscr{L} A_k(\zeta, 1) \\ &\equiv \eta a(\eta)\, A_k(\zeta, 1) - \epsilon a(\eta)\, A_k(\zeta, -1) + \epsilon^2 c(\eta)\, A_k(\zeta, 0). \end{aligned} \tag{4.3}$$

In this approximation the relative errors associated with V_1 are $O(\epsilon^2, \epsilon^{\frac{9}{5}}, \epsilon^{\frac{3}{2}})$ for $\eta \in \mathbf{T}_2 \cup \mathbf{T}_3$. To assess the errors associated with this approximation to V_1 for $\eta \in \mathbf{T}_1$ we first sum equation (4.3) over k to obtain

$$\epsilon V_1(\eta) \sim -\mathscr{L} A_3(\zeta, 2) - U_0^{(0)}(\eta) - \mathscr{L} A_2(\zeta, 2). \tag{4.4}$$

On comparing this approximation with the exact connection formula satisfied by the V_k (Reid 1972, eqn (4.2)) it is found that the relative errors associated with the dominant and recessive terms in the approximation are both $O(\epsilon^2, \epsilon^{\frac{9}{5}}, \epsilon^{\frac{3}{2}})$. As might be expected, however, the balanced term has relative errors $O(\epsilon^2, \epsilon^{\frac{12}{5}}, \epsilon^3)$. The corresponding approximations to V_k' are given by

$$\begin{aligned} \epsilon V_k'(\eta) &\sim \mathscr{M} A_k(\zeta, 1) \\ &\equiv (\eta a)' A_k(\zeta, 1) - \epsilon(a' - c)\, A_k(\zeta, -1) + \epsilon^2(\mathscr{b} + c')\, A_k(\zeta, 0). \end{aligned} \tag{4.5}$$

The relative errors associated with this approximation to V_1' are $O(\epsilon^3 \ln \epsilon, \epsilon^3, \epsilon^3)$ for $\eta \in \mathbf{T}_2 \cup \mathbf{T}_3$ and, on summing over k, they are found to remain of the same orders for $\eta \in \mathbf{T}_1$.

Consider next the solutions of balanced type for which the corresponding approximations are

$$U_k(\eta) \sim \mathscr{G}(\eta) - \eta \mathscr{A}(\eta) - (U_c''/U_c')\, \mathscr{L} B_k(\zeta, 1, 1) \tag{4.6}$$

and

$$U_k'(\eta) \sim \mathscr{G}'(\eta) - \eta \mathscr{A}'(\eta) - (U_c''/U_c')\, \mathscr{M} B_k(\zeta, 1, 1). \tag{4.7}$$

For $\eta \in \mathbf{T}_k$, the errors associated with these approximations are both $O(\epsilon^3 \ln \epsilon, \epsilon^3, \epsilon^3)$. To estimate the errors in the complementary sectors $\mathbf{I} \backslash \mathbf{T}_k$ it is again necessary to use the connection formulae. For example, on using the connection formula

$$B_3(\zeta, 1, 1) = B_1(\zeta, 1, 1) + 2\pi \mathrm{i}\{1 + A_2(\zeta, 1)\} \tag{4.8}$$

we have

$$U_3(\eta) \sim \mathscr{G}(\eta) - (1 + 2\pi\mathrm{i})\, \eta \mathscr{A}(\eta) - (U_c''/U_c')\, \mathscr{L} B_1(\zeta, 1, 1) - 2\pi\mathrm{i}(U_c''/U_c')\, \mathscr{L} A_2(\zeta, 1). \tag{4.9}$$

Thus, for $\eta \in \mathbf{T}_1$, the errors associated with the balanced term are again $O(\epsilon^3 \ln \epsilon, \epsilon^3, \epsilon^3)$ but for the term of dominant-recessive type they are $O(\epsilon^2, \epsilon^{\frac{9}{5}}, \epsilon^{\frac{3}{2}})$.

We must also consider the errors involved in approximating $A(\alpha, c, \epsilon)$ by $A^{(0)}(\alpha, c)$. For symmetrical flows in a channel it is known that

$$A^{(0)}(\alpha, c) = U_c'^2 \alpha^{-2} \left[\int_{z_c}^{z_2} (U - c)^2 \, \mathrm{d}z \right]^{-1} + O(1) \quad \text{as} \quad \alpha \downarrow 0 \tag{4.10}$$

and it is not difficult to show that $A^{(1)}(\alpha, c) = O(\alpha^{-4})$ as $\alpha \downarrow 0$. Accordingly, $A^{(0)}(\alpha, c)$ provides an approximation to $A(\alpha, c, \epsilon)$ with relative errors $O(\epsilon^2, \epsilon^{\frac{12}{5}}, \epsilon^3)$. From equation (4.10) it also follows that the product $\eta A^{(0)}(\alpha, c)$ is of order one along the entire curve of marginal stability.

The disparity between these various error estimates emphasizes once again the distinction which must be drawn between approximations to the solutions of the Orr–Sommerfeld equation on the one hand and approximations to the eigenvalue relation on the other.

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5. The first approximation to the eigenvalue relation

We now wish to derive a first approximation to $\Delta(\alpha, c, \epsilon; z)$ which is uniformly valid in a full neighbourhood of z_c. For that purpose it is convenient to rewrite equation (2.27) in the equivalent form

$$\Delta(z) = \eta'(z)\, \mathscr{W}(\Psi, \epsilon V_1)(\eta), \tag{5.1}$$

where

$$\Psi(\eta) = AU_0(\eta) + U_3(\eta) \tag{5.2}$$

and $A(\alpha, c, \epsilon)$ is still given by (2.21). Suppose now that we approximate V_1 and V_1' by equations (3.16) and (4.5) but temporarily regard the remaining terms as exact. Then we obtain

$$\Delta(z) \sim \eta'\{[\Psi(\eta a)' - \Psi'(\eta a + \epsilon^3 b)]\, A_1(\zeta, 1) \\ - \epsilon[\Psi(a' - c) - \Psi' a]\, A_1(\zeta, -1) + \epsilon^2[\Psi(b + c') - \Psi' c]\, A_1(\zeta, 0)\}. \tag{5.3}$$

Clearly we can neglect $\epsilon^3 b$ compared to ηa with relative errors $O(\epsilon^2, \epsilon^{\frac{12}{5}}, \epsilon^3)$ and this is equivalent to approximating V_1 by equation (4.3) rather than equation (3.16). A similar argument shows that in a first approximation to Δ we must also approximate U_3 by equation (4.6) rather than equation (3.15). Thus, on approximating A, U_0 and U_3 we obtain

$$\begin{aligned}\Delta(z) \sim \eta'\{&[(\mathscr{G} - \eta\mathscr{A})(\eta a)' - (\mathscr{G}' - \eta\mathscr{A}')\,\eta a]\, A_1(\zeta, 1) \\ &- \epsilon[(A^{(0)}U_0^{(0)} + \mathscr{G} - \eta\mathscr{A})(a' - c) - (A^{(0)}U_0^{(0)\prime} + \mathscr{G}' - \eta\mathscr{A}')\, a]\, A_1(\zeta, -1) \\ &+ \epsilon^2[(A^{(0)}U_0^{(0)} + \mathscr{G} - \eta\mathscr{A})(b + c') - (A^{(0)}U_0^{(0)\prime} + \mathscr{G}' - \eta\mathscr{A}')\, c]\, A_1(\zeta, 0) \\ &+ \epsilon\mathscr{A}(a + \eta c)\frac{d}{d\zeta}\mathscr{W}[A_1(\zeta, 1), B_3(\zeta, 1, 1)] \\ &- \epsilon^2[\mathscr{C}(\eta\alpha)' - (\mathscr{B} + \mathscr{C}')\,\eta a]\, \mathscr{W}[A_1(\zeta, 1), B_3(\zeta, 1, 1)] \\ &+ \epsilon^3[\mathscr{A}(b + c') - (\mathscr{A}' - \mathscr{C})\, c]\, \mathscr{W}[A_1(\zeta, 0), B_3(\zeta, 0, 1)]\}.\end{aligned} \tag{5.4}$$

A crucial step in the simplification of this result is the recognition of the fact that it is possible to eliminate the B-type Airy functions from equation (5.4). For this purpose let

$$w(\zeta) = \mathscr{W}[A_1(\zeta, 1), B_3(\zeta, 1, 1)]. \tag{5.5}$$

Then it is easy to show that w satisfies the inhomogeneous equation

$$d^3w/d\zeta^3 - \zeta\, dw/d\zeta - w = 2A_1(\zeta, 0). \tag{5.6}$$

Since w is recessive in the sector $\mathbf{S}_1 \cap \mathbf{T}_3$ the general solution of this equation must be of the form

$$w(\zeta) = 2A_1(\zeta, 0, 1) + CA_1(\zeta, 0). \tag{5.7}$$

The constant in this equation can easily be evaluated by setting $\zeta = 0$ and using the initial values

$$\left.\begin{aligned} A_1(0, 1) &= -\tfrac{1}{3}, \\ A_1(0, 0) &= 3^{-\frac{2}{3}}/\Gamma(\tfrac{2}{3}) \\ A_1(0, 0, 1) &= \frac{1}{3}\left(-\gamma + \frac{\pi}{2\sqrt{3}} - \tfrac{1}{2}\ln 3 + 3\pi i\right) A_1(0, 0), \\ B_3(0, 0, 1) &= -\frac{2\pi}{\sqrt{3}}\, e^{\frac{1}{3}i\pi} A_1(0, 0), \\ \text{and}\quad B_3(0, 1, 1) &= \tfrac{1}{3}(-\gamma + \ln 3 + 7\pi i), \end{aligned}\right\} \tag{5.8}$$

where γ is Euler's constant. In this way we obtain

$$\mathscr{W}[A_1(\zeta,1), B_3(\zeta,1,1)] = 2A_1(\zeta,0,1) + (\gamma - 4\pi \mathrm{i})\, A_1(\zeta,0). \tag{5.9}$$

A similar calculation also gives

$$\mathscr{W}[A_1(\zeta,0), B_3(\zeta,0,1)] = A_1(\zeta,1). \tag{5.10}$$

Thus, in a first approximation to Δ, the terms in equation (5.4) which are formally of order ϵ^3 can also be neglected. The coefficient of $A_1(\zeta,1)$ can be substantially simplified by noting that

$$(\mathscr{G} - \eta\mathscr{A})\,(\eta a)' - (\mathscr{G}' - \eta\mathscr{A}')\,\eta a \equiv -\eta'^{-1}\mathscr{W}(\phi_1^{(0)}, \phi_2^{(0)})\,(z) = \eta'^{-1}. \tag{5.11}$$

For computational purposes it is convenient to express the remaining coefficients, so far as possible in terms of z. If, as usual, we let

$$\Phi^{(0)}(z) = A^{(0)}(\alpha, c)\,\phi_1^{(0)}(z) + \phi_2^{(0)}(z) \tag{5.12}$$

then, after some reduction, we obtain

$$\begin{aligned}\Delta(z) \sim A_1(\zeta,1) &+ \epsilon\eta^{-1}\eta'^{-\frac{3}{2}}\{[\Phi^{(0)}(z) - \eta'^{\frac{3}{2}}]\,A_1(\zeta,-1)\\ &+ (U_c''/U_c')\,\phi_1^{(0)}(z)\,[2A_1(\zeta,-1,1) - (\ln\zeta + 2\pi\mathrm{i})\,A_1(\zeta,-1)]\}\\ &+ \epsilon^2\{[\eta'\Phi^{(0)}(z)\,(b + c') - \Phi^{(0)\prime}(z)\,c]\,A_1(\zeta,0) + (U_c''/U_c')\,[\eta'\phi_1^{(0)}(z)\,(b + c') - \phi_1^{(0)\prime}(z)\,c]\\ &\quad \times [2A_1(\zeta,0,1) - (\ln\zeta + 2\pi\mathrm{i})\,A_1(\zeta,0)]\}.\end{aligned} \tag{5.13}$$

We also note that for any function $f(z)$ we have

$$\begin{aligned}\eta' f(z)\,(b + c') - f'(z)\,c &= \eta^{-2}\mathscr{W}(\phi_1^{(0)}, f)\,(z)\\ &- \eta^{\frac{1}{4}}\left(\frac{U-c}{U_c'}\right)^{-\frac{3}{4}}\left\{\left(\frac{U-c}{U_c'}\right)^{-\frac{1}{2}}\left[f'(z) + \frac{5}{4}\frac{U'}{U-c}f(z)\right] - G_1(z)f(z) - \tfrac{7}{48}\eta^{-\frac{3}{2}}f(z)\right\}.\end{aligned} \tag{5.14}$$

Thus, the slowly varying coefficients in equation (5.13) can all be expressed in terms of the Langer variable $\eta(z)$, the basic velocity distribution $U(z)$, the solutions of the inviscid equation $\phi_1^{(0)}(z)$ and $\phi_2^{(0)}(z)$, and the coefficient $G_1(z)$ which appears in the outer expansion of $\phi_3(z)$.

Equation (5.13) provides a 'first approximation' to $\Delta(z)$ with relative errors $O(\epsilon^2 \ln\epsilon, \epsilon^{\frac{12}{5}}, \epsilon^3)$ and further simplification is not possible without destroying the uniformity of the approximation. The form in which we have written equation (5.13) shows that the terms

$$2A_1(\zeta,p,1) - (\ln\zeta + 2\pi\mathrm{i})\,A_1(\zeta,p) \quad (p = -1, 0) \tag{5.15}$$

play a rôle very similar to the 'viscous corrections' introduced by Tollmein (1947) in his discussion of the singular inviscid solution.

6. An alternate derivation of the first approximation to the eigenvalue relation

Second-order Wronskians clearly play an important role both in the formulation of the eigenvalue problem and in the simplification of equation (5.4). In this section, therefore, we wish to show that if u and v denote any two solutions of the transformed equation (3.3) then it is possible to derive a pair of coupled third-order equations for $\mathscr{W}(u, v)\,(\eta)$ and $\mathscr{W}(u', v')\,(\eta)$ from which we can obtain a simplified derivation of the approximation (5.13). The method which will be used here is similar to the one described by Gilbert & Backus (1966) in their discussion of propagator matrices for elastic wave problems.

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If we adopt the vector notation $\boldsymbol{u} = (u, u', u'', u''',)^{\mathrm{T}}$ for column vectors then equation (3.3) becomes

$$\chi' = \boldsymbol{M}\chi, \tag{6.1}$$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon^{-3}h_0 + h_1 & \epsilon^{-3}g_0 + g_1 & \epsilon^{-3}\eta + f_1 & -f_0 \end{bmatrix}. \tag{6.2}$$

Now let $u_i = w_i^1$ and $v_i = w_i^2$ $(i = 1, 2, 3, 4)$ be any two solutions of equation (6.1). The 2×2 minors of the matrix w_i^r $(i = 1, 2, 3, 4;\ r = 1, 2)$ are given by

$$W_{ij} = \epsilon_{rs}\, w_i^r\, w_j^s \tag{6.3}$$

where
$$\epsilon_{rs} = \begin{cases} 1 & \text{if} \quad r = 1 \quad \text{and} \quad s = 2 \\ -1 & \text{if} \quad r = 2 \quad \text{and} \quad s = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{6.4}$$

and a simple calculation shows that they satisfy the equation

$$W'_{ij} = M_{ik}\, W_{kj} - M_{jk}\, W_{ki}. \tag{6.5}$$

The six independent components of this equation are

$$\left.\begin{aligned} W'_{12} = W_{13}, \quad W'_{13} = W_{14} + W_{23}, \quad W'_{23} = W_{24}, \\ W'_{14} + f_0\, W_{14} = W_{24} + (\epsilon^{-3}g_0 + g_1)\, W_{12} + (\epsilon^{-3}\eta + f_1)\, W_{13}, \\ W'_{24} + f_0\, W_{24} = W_{34} - (\epsilon^{-3}h_0 + h_1)\, W_{12} + (\epsilon^{-3}\eta + f_1)\, W_{23} \\ \text{and} \qquad W'_{34} + f_0\, W_{34} = -(\epsilon^{-3}h_0 + h_1)\, W_{13} - (\epsilon^{-3}g_0 + g_1)\, W_{23}. \end{aligned}\right\} \tag{6.6}$$

On eliminating W_{13}, W_{14}, W_{24} and W_{34} we obtain the pair of coupled third-order equations

$$\epsilon^3(W'''_{12} + f_0\, W''_{12}) - (\eta + \epsilon^3 f_1)\, W'_{12} - (g_0 + \epsilon^3 g_1)\, W_{12} = \epsilon^3(2W'_{23} + f_0\, W_{23}) \tag{6.7}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}\eta} + f_0\right)\{\epsilon^3(W''_{23} + f_0\, W'_{23}) - (\eta + \epsilon^3 f_1)\, W_{23}\} + (g_0 + \epsilon^3 g_1)\, W_{23}$$
$$= -\left(\frac{\mathrm{d}}{\mathrm{d}\eta} + f_0\right)(h_0 + \epsilon^3 h_1)\, W_{12} - (h_0 + \epsilon^3 h_1)\, W'_{12}. \tag{6.8}$$

A general discussion of these equations lies outside the scope of the present paper but we do wish to show how they can be used to provide an alternate and much simpler derivation of the approximation (5.13).

Suppose now that we identify u with $\Psi(\equiv AU_0 + U_3)$ and v with ϵV_1. Then

$$W_{12} = \mathscr{W}(\Psi, \epsilon V_1)\,(\eta) \quad \text{and} \quad W_{23} = \mathscr{W}(\Psi', \epsilon V'_1)\,(\eta). \tag{6.9}$$

Since W_{12} and W_{23} are both recessive for finite values of η in $\mathbf{S}_1 \cap \mathbf{T}_3$, this suggests, though it has not yet been proved, that the rapidly varying terms in their uniform expansions can be expressed to all orders in terms of $A_1(\zeta, p, q)$ $(p = 0, \pm 1;\ q = 0, 1, 2, \ldots)$. A preliminary study of equations (6.7) and (6.8), however, shows that the first approximations to $\mathscr{W}(\Psi, \epsilon V_1)$ and $\mathscr{W}(\Psi', \epsilon V'_1)$ must have the forms

$$\mathscr{W}(\Psi, \epsilon V_1)\,(\eta) \sim C_1(\eta)\, A_1(\zeta, 1) + \epsilon\{C_2(\eta)\, A_1(\zeta, -1, 1) + C_3(\eta)\, A_1(\zeta, -1)\}$$
$$+ \epsilon^2\{C_4(\eta)\, A_1(\zeta, 0, 1) + C_5(\eta)\, A_1(\zeta, 0)\} \tag{6.10}$$

and

$$\mathscr{W}(\Psi', \epsilon V_1') \sim \epsilon^{-1}\{D_1(\eta)\, A_1(\zeta, 0, 1) + D_2(\eta)\, A_1(\zeta, 0)\} + D_3(\eta)\, A_1(\zeta, 1) + \epsilon\{D_4(\eta)\, A_1(\zeta, -1, 1) + D_5(\eta)\, A_1(\zeta, -1)\}. \quad (6.11)$$

The slowly varying terms in these approximations can be determined, at least in principle, by deriving and then solving the differential equations which they satisfy. For example, it is easy to show that

$$\eta C_1' + g_0(\eta)\, C_1 = 0 \quad (6.12)$$

and hence that

$$C_1(\eta) = \eta'^{-1}, \quad (6.13)$$

where the constant of integration has been fixed by requiring that $C_1(0) = 1$, but the remaining nine coefficients satisfy coupled differential equations and their determination by this method is somewhat complicated. Fortunately, however, once $C_1(\eta)$ is known, $C_2(\eta), \ldots, C_4(\eta)$ can be found quite easily by the use of a simple matching technique. For this purpose we need the outer expansion of $\mathscr{W}(\Phi, \epsilon\phi_3)(z)$ for η in $\mathbf{T}_2 \cup \mathbf{T}_3$. On using equation (2.10) we obtain

$$\mathscr{W}(\Phi, \epsilon\phi_3)(z) \sim -\tfrac{1}{2}\pi^{-\frac{1}{2}}\epsilon^{\frac{1}{4}}\left(\frac{U-c}{U_c'}\right)^{-\frac{3}{4}} \exp\{-\epsilon^{-\frac{3}{2}}Q(z)\}$$
$$\times\left\{\Phi^{(0)} - \epsilon^{\frac{3}{2}}\left[G_1(z)\,\Phi^{(0)} - \left(\frac{U-c}{U_c'}\right)^{-\frac{1}{2}}\left(\Phi^{(0)\prime} + \frac{5}{4}\frac{U'}{U-c}\Phi^{(0)}\right)\right] + O(\epsilon^3)\right\} \quad (\eta \in \mathbf{T}_2 \cup \mathbf{T}_3). \quad (6.14)$$

Similarly, on taking the outer expansion of equation (6.10) we have

$$\mathscr{W}(\Psi, \epsilon V_1)(\eta) \sim -\tfrac{1}{2}\pi^{-\frac{1}{2}}\zeta^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}\zeta^{\frac{3}{2}}\right)\{C_1 + \eta(\tfrac{1}{2}\ln\zeta + \pi i)\, C_2 + \eta C_3$$
$$- \epsilon^{\frac{3}{2}}[\tfrac{41}{48}\eta^{-\frac{3}{2}}C_1 - \tfrac{7}{48}(\tfrac{1}{2}\ln\zeta + \pi i)\,\eta^{-\frac{1}{2}}C_2 - \tfrac{7}{48}\eta^{-\frac{1}{2}}C_3 + (\tfrac{1}{2}\ln\zeta + \pi i)\,\eta^{\frac{1}{2}}C_4 + \eta^{\frac{1}{2}}C_5] + O(\epsilon^3)\}. \quad (6.15)$$

Since $\mathscr{W}(\Phi, \epsilon\phi_3) = \eta'\mathscr{W}(\Psi, \epsilon V_1)$ we see that, to first order, we must have

$$\eta'\{C_1 + \eta(\tfrac{1}{2}\ln\zeta + \pi i)\, C_2 + \eta C_3\} = \eta'^{-\frac{3}{2}}\Phi^{(0)}(z), \quad (6.16)$$

where we have used the relations $\eta\eta'^2 = (U-c)/U_c'$ and $Q(z) = \frac{2}{3}\eta^{\frac{3}{2}}$. Since C_1 is known from equation (6.13) and C_2 and C_3 must both be analytic at $\eta = 0$, we see immediately that

$$\eta' C_2(\eta) = 2(U_c''/U_c')\,\eta^{-1}\eta'^{-\frac{3}{2}}\phi_1^{(0)}(z) \quad (6.17)$$

and

$$\eta' C_3(\eta) = \eta^{-1}\eta'^{-\frac{3}{2}}\{\Phi^{(0)}(z) - \eta'^{\frac{3}{2}} - (U_c''/U_c')\,(\ln\zeta + 2\pi i)\,\phi_1^{(0)}(z)\}. \quad (6.18)$$

Similarly, to order $\epsilon^{\frac{3}{2}}$, we have

$$\eta'\{(\tfrac{1}{2}\ln\zeta + \pi i)\, C_4 + C_5\} = \eta'\Phi^{(0)}(z)\,(\mathscr{b} + \mathscr{c}') - \Phi^{(0)\prime}(z)\,\mathscr{c}, \quad (6.19)$$

where equation (5.14) has been used with $f(z) = \Phi^{(0)}(z)$, and this gives

$$\eta' C_4(\eta) = 2(U_c''/U_c')\,\{\eta'\phi_1^{(0)}(z)\,(\mathscr{b} + \mathscr{c}') - \phi_1^{(0)\prime}(z)\,\mathscr{c}\} \quad (6.20)$$

and

$$\eta' C_5(\eta) = \eta'\Phi^{(0)}(z)\,(\mathscr{b} + \mathscr{c}') - \Phi^{(0)\prime}(z)\,\mathscr{c} - (U_c''/U_c')\,(\ln\zeta + 2\pi i)\,\{\eta'\phi_1^{(0)}(z)\,(\mathscr{b} + \mathscr{c}') - \phi_1^{(0)\prime}(z)\,\mathscr{c}\}. \quad (6.21)$$

These results are seen to be in complete agreement with equation (5.13). Thus, once the general structure of the uniform expansion of $\Delta(z)$ is known, the slowly varying terms in it can be obtained relatively easily by this matching technique.

7. A computational form of the first approximation to the eigenvalue relation

The first approximation to the eigenvalue relation is obtained by simply setting $z = z_1$ in equation (5.13) and then equating the right hand side of the equation to zero. The Airy functions which appear in this approximation are all rapidly varying functions of $\zeta_1 = \eta(z_1)/\epsilon$ and this behaviour can cause numerical difficulties. Such difficulties can be entirely avoided, however, if equation (5.13) is divided through by $A_1(\zeta_1, 0)$ (say). The resulting ratios of Airy functions can then be expressed in terms of certain Tietjens-type functions which are easy to compute. Consider then the generalized Tietjens function

$$F(Z, p) = \frac{A_1(\zeta_1, p)}{\zeta_1 A_1(\zeta_1, p-1)}, \tag{7.1}$$

where

$$\zeta_1 = Z\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}} \text{ and } Z = \left[\frac{3}{2}\int_{z_1}^{z_c} |U-c|^{\frac{1}{2}}\,\mathrm{d}z\right]^{\frac{2}{3}} (\alpha R)^{\frac{1}{3}}, \tag{7.2}$$

which were first introduced by Hughes & Reid (1968). When $p = 2$ and 0 this gives the ordinary and adjoint Tietjens functions $F(Z)$ and $F^{\dagger}(Z)$ respectively both of which have been extensively tabulated. For the present purposes, however, it is more convenient to let

$$H(Z, p) = ZF(Z, p)$$

$$= \mathrm{e}^{\frac{5}{6}\pi\mathrm{i}} \frac{A_1(\zeta_1, p)}{A_1(\zeta_1, p-1)} \tag{7.3}$$

and to define the related functions

$$K(Z, p) = \frac{A_1(\zeta_1, p, 1)}{A_1(\zeta_1, p)}. \tag{7.4}$$

The first approximation to the eigenvalue relation can then be written in the form

$$\begin{aligned}
[\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}/A_1(\zeta_1, 0)]\,\Delta(z_1) \sim{} & H(Z, 1) + \eta'^{-\frac{3}{2}}\{\Phi^{(0)}(z_1) - \eta'^{\frac{3}{2}} \\
& + (U_c''/U_c')\,\phi_1^{(0)}(z_1)\,[2K(Z, -1) - (\ln Z + \tfrac{7}{6}\pi\mathrm{i})]\}\,H(Z, -1) \\
& + \epsilon^2\,\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}\{\eta'\Phi^{(0)}(z_1)\,(\mathscr{b} + \mathscr{c}') - \Phi^{(0)\prime}(z_1)\,\mathscr{c} \\
& + (U_c''/U_c')\,[\eta'\phi_1^{(0)}(z_1)\,(\mathscr{b} + \mathscr{c}') - \phi_1^{(0)\prime}\mathscr{c}] \\
& \times [2K(Z, 0) - (\ln Z + \tfrac{7}{6}\pi\mathrm{i})]\} = 0,
\end{aligned} \tag{7.5}$$

where it is understood that η' is to be evaluated at z_1 and $\mathscr{b}$, $\mathscr{c}$ and $\mathscr{c}'$ are to be evaluated at

$$\eta_1 \equiv \eta(z_1).$$

The Tietjens-type functions which appear in equation (7.5) are particularly easy to compute. We first note that

$$H(Z, -1) = \mathrm{i}/\{ZH(Z, 0)\} \tag{7.6}$$

and that $H(Z, 0)$ satisfies the first-order *nonlinear* equation

$$H'(Z, 0) - \mathrm{i}Z\{H(Z, 0)\}^2 = 1 \tag{7.7}$$

and the initial condition

$$H(0, 0) = 3^{-\frac{1}{3}}\{\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{2}{3})\}\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}. \tag{7.8}$$

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Once $H(Z,0)$ has been computed, we can obtain $H(Z,1)$ as the solution of the first-order *linear* equation

$$H'(Z,1)+\{H(Z,0)\}^{-1}H(Z,1)=1 \tag{7.9}$$

which satisfies the initial condition

$$H(0,1)=3^{-\frac{1}{3}}\Gamma(\tfrac{2}{3})\,\mathrm{e}^{-\frac{1}{6}\pi \mathrm{i}}.$$

It is also easy to show that

$$K(Z,0)=-\tfrac{1}{2}\mathrm{i}\int_0^Z\{H(Z',1)\}^2\mathrm{d}Z'+\frac{1}{3}\left(-\gamma+\frac{\pi}{2\sqrt{3}}-\tfrac{1}{2}\ln 3+3\pi \mathrm{i}\right) \tag{7.10}$$

and

$$K(Z,-1)=K(Z,0)-\tfrac{1}{2}\mathrm{i}H(Z,0)\{H(Z,1)\}^2. \tag{7.11}$$

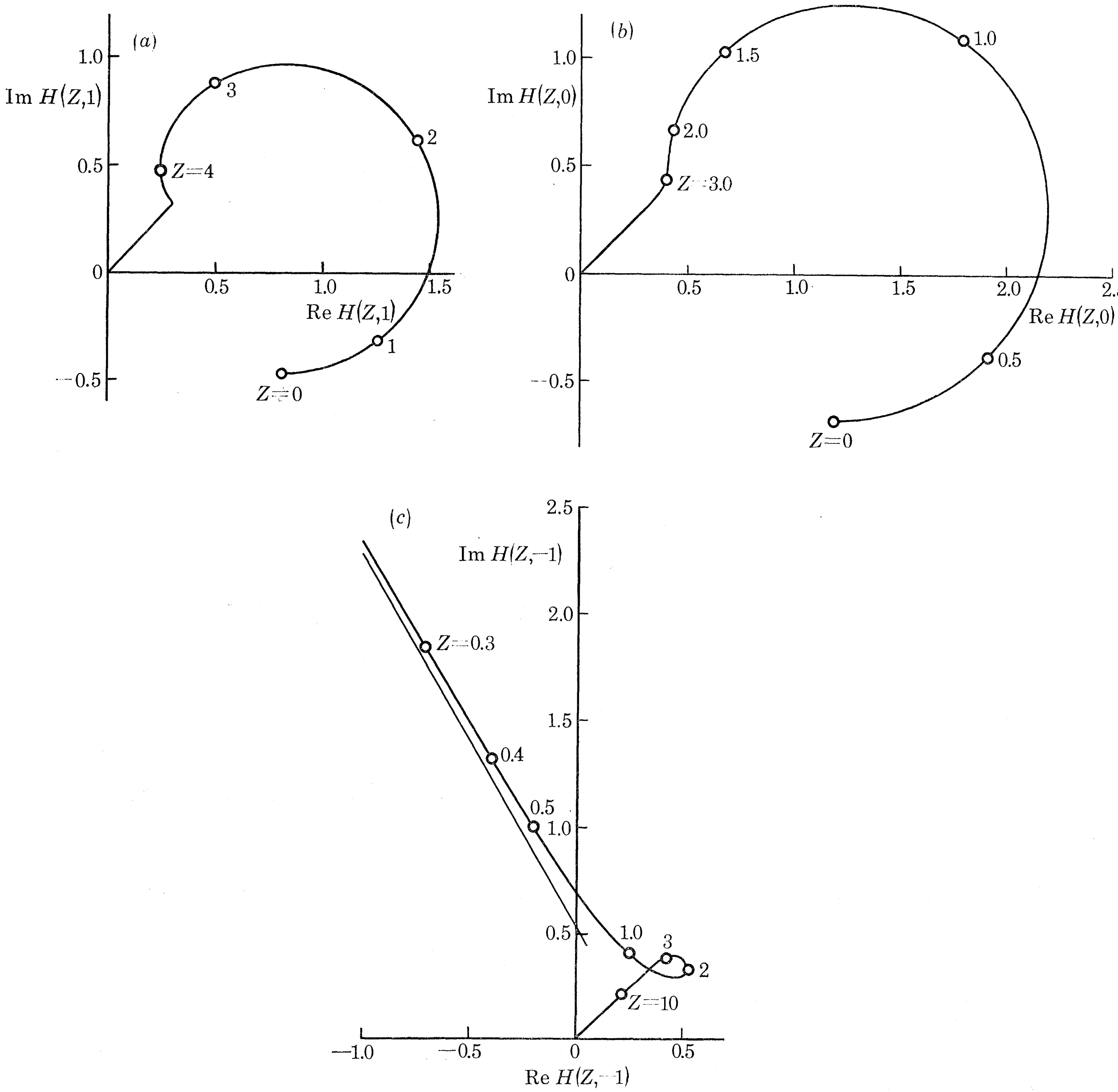

FIGURE 2. The behaviour of $H(Z,p)$ for $p=0,\pm 1$. In (c) the asymptote as $Z\to 0$ is $\sqrt{3}\,H_r(Z,-1)+H_i(Z,-1)\simeq 0.531$.

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Thus, the computation of the four Tietjens-type functions which appear in equation (7.5) require only the integration of two first-order equations and one quadrature. The general behaviour of these functions is shown in figures 2 and 3. We also note that

$$H(Z,p) \sim e^{\frac{1}{4}\pi i}Z^{-\frac{1}{2}}\{1+\tfrac{1}{4}(2p+1)\,e^{\frac{1}{4}\pi i}Z^{-\frac{3}{2}}+O(Z^{-3})\} \tag{7.12}$$

and

$$K(Z,p) \sim \tfrac{1}{2}\ln Z+\tfrac{7}{12}\pi i-\tfrac{1}{48}(24p-17)\,e^{\frac{1}{4}\pi i}Z^{-\frac{3}{2}}+O(Z^{-3}). \tag{7.13}$$

These expansions are valid in the complete sense of Watson in the sector $\frac{1}{6}\pi < \mathrm{ph}\,Z < \frac{3}{2}\pi$ but they remain valid in the sense of Poincaré in the larger sector $-\frac{1}{6}\pi < \mathrm{ph}\,Z < \frac{11}{6}\pi$.

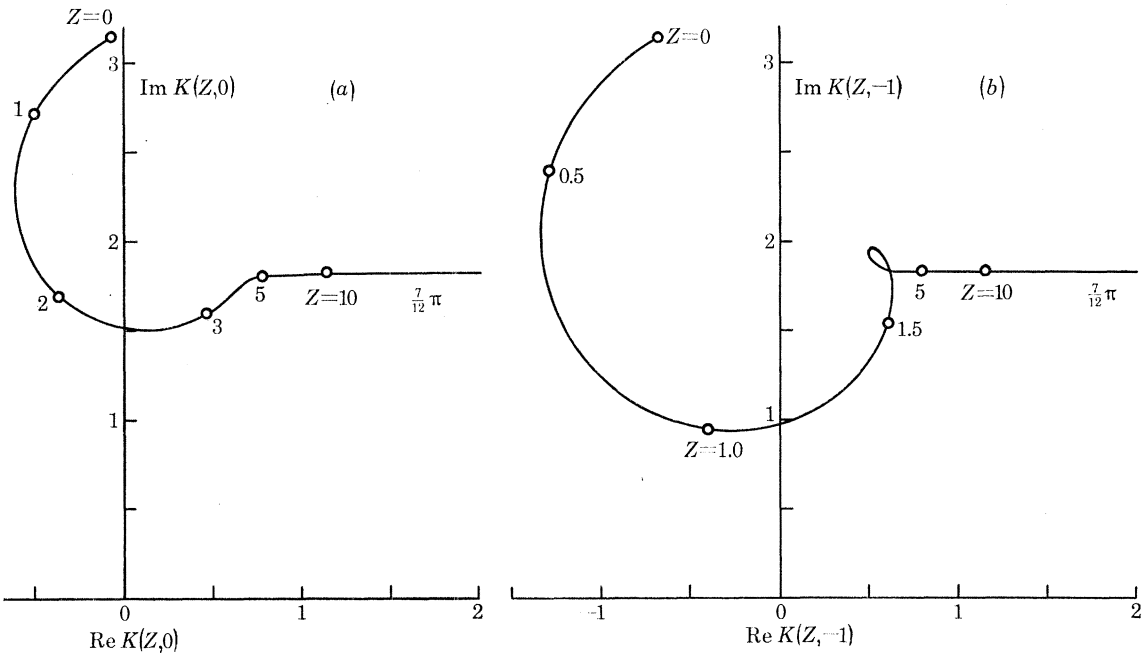

FIGURE 3. The behaviour of $K(Z,p)$ for $p = 0$ and -1.

To illustrate the accuracy of the approximation (7.5), we have made a calculation of the curve of marginal stability for plane Poiseuille flow. For this flow we have $U(z) = 1-z^2$, $z_1 = -1$, $z_c = -(1-c)^{\frac{1}{2}}$ and $z_2 = 0$. We also have

$$Z = \{\tfrac{3}{4}[\sqrt{c}-(1-c)\,\mathrm{artanh}\,\sqrt{c}]\}^{\frac{2}{3}}(\alpha R)^{\frac{1}{3}} \tag{7.14}$$

and

$$G_1(z_1) = -i2^{\frac{1}{2}}(1-c)^{\frac{1}{4}}\left\{\frac{101}{24}c^{-\frac{3}{2}}+\frac{23}{24}c^{-\frac{1}{2}}+\frac{23}{24}\frac{c^{\frac{1}{2}}}{1-c}+\tfrac{1}{2}\alpha^2\,\mathrm{artanh}\,\sqrt{c}\right\}. \tag{7.15}$$

The computational procedure used to solve equation (7.5) was similar to the one described by Hughes & Reid (1968) and need not be repeated again here. The values of the parameters associated with the minimum critical Reynolds number were found to be†

$$R_c = 5769.7, \quad \alpha_c = 1.0207, \quad \text{and} \quad c = 0.2640, \tag{7.16}$$

† In calculating the curve of marginal stability it is found convenient to fix Z and iterate for α, c and R. Thus, the calculations were made for $Z = 2.325\,(0.025)\,2.925\,(0.0025)\,3.275\,(0.025)\,8$ which corresponds to $R_c \leqslant R \lesssim 6\times10^8$ along the upper branch and $R_c \leqslant R \lesssim 4\times10^7$ along the lower branch. These remarks also apply to the results described in §8. Copies of these results with some further details of the computational procedure can be obtained from B. S. Ng.

and these results should be compared with the 'exact' numerical values

$$R_c = 5772.22, \quad \alpha_c = 1.0205, \quad \text{and} \quad c = 0.2640 \tag{7.17}$$

which were obtained by Orszag (1971) and subsequently confirmed by Davey (unpublished). The values of α and c for $R = 7500$ and 9000 have also been computed numerically by Reynolds

Table 2. A comparison with the results of Reynolds & Potter (1967)

		uniform approximation		Reynolds & Potter	
R		α	c	α	c
7500	lower branch	0.8750	0.2345	0.875	0.2344
	upper branch	1.0944	0.2597	1.094	0.2597
9000	lower branch	0.8233	0.2203	0.823	0.2203
	upper branch	1.0971	0.2515	1.097	0.2515

& Potter (1967) and a comparison with their results is given in table 2. Near the nose of the curve of marginal stability the error associated with the approximation (7.5) would be expected to be of the order of ϵ^3. The worst error in the present results is in the value of R_c for which the actual error is about 0.044 % compared with an expected error of about 0.017%.

It is easy to verify that equation (7.5) gives the correct behaviour of the asymptotes to the upper and lower branches of the curve of marginal stability as $R \to \infty$ and that the errors are then of order $\epsilon^{\frac{12}{5}}$ and $\epsilon^2 \ln \epsilon$ respectively. Thus, equation (7.5) provides a uniform 'first approximation' to the eigenvalue relation which is valid along the entire curve of marginal stability.

8. Heuristic approximations to the eigenvalue relation

The approximation (7.5) has a very different structure from the so-called heursitic approximations which have been widely used in the past and it is of some interest, therefore, to compare the results obtained by using these heuristic approximations with those obtained by using the uniform approximation (7.5). In the heuristic approach to the eigenvalue problem, ϕ_1 and ϕ_2 are approximated by the first terms in their outer expansions and various approximations to ϕ_3 are then considered. Thus we approximate equation (2.27) by

$$\Delta(z) \approx \Delta^{(0)}(z) \equiv \mathscr{W}(\Phi^{(0)}, \epsilon\phi_3)(z). \tag{8.1}$$

(a) *The local turning point approximation to* $\phi_3(z)$

If $\phi_3(z)$ is approximated by the first term of its inner expansion then we have

$$\phi_3(z) \sim A_1(\xi, 2), \quad \text{where} \quad \xi = (z-z_c)/\epsilon, \tag{8.2}$$

and equation (8.1) becomes

$$\Delta^{(0)}(z) \sim \{\Phi^{(0)}(z) - (z-z_c)\,\Phi^{(0)\prime}(z)\}\,A_1(\xi, 1) + \epsilon\Phi^{(0)\prime}(z)\,A_1(\xi, -1). \tag{8.3}$$

For computational purposes this is usually written in the form

$$\frac{U_1'}{c}\frac{\Phi^{(0)}(z_1)}{\Phi^{(0)\prime}(z_1)} + \{1+\lambda_1(c)\}F(Y) = 0, \tag{8.4}$$

where

$$\left.\begin{aligned} \xi_1 &= Y e^{-\frac{5}{6}\pi i}, \quad Y = (z_c - z_1)\,(\alpha R U_c')^{\frac{1}{3}}, \\ 1+\lambda_1(c) &= (U_1'/c)\,(z_c - z_1) \end{aligned}\right\} \tag{8.5}$$

and $F(Y)$ is the Tietjens function [cf. equation (7.1)]. Calculations based on equation (8.4) lead to the results given in table 3 and shown in figure 4. In this approximation, the asymptotes to the upper and lower branches of the curve of marginal stability are given correctly but the errors are of order $\epsilon^{\frac{6}{5}}$ and $\epsilon \ln \epsilon$ respectively. The approximation is clearly not uniform, however, since it involves the outer expansions of ϕ_1 and ϕ_2 combined with the inner expansion of ϕ_3.

Table 3. Results for plane Poiseuille flow based on the heuristic approximations to the eigenvalue relation

approximation to $\phi_3(z)$	equation	R_c	α_c	c
local turning point	(8.3)	5397.1	1.022	0.2672
Tollmien	(8.8)	5697.3	1.010	0.2607
uniform (truncated equation)	(8.14)	4880.9	1.034	0.2721
uniform (Orr–Sommerfeld equation)	(8.15)	6052.1	1.020	0.2621
'exact' values	—	5772.2	1.021	0.2640

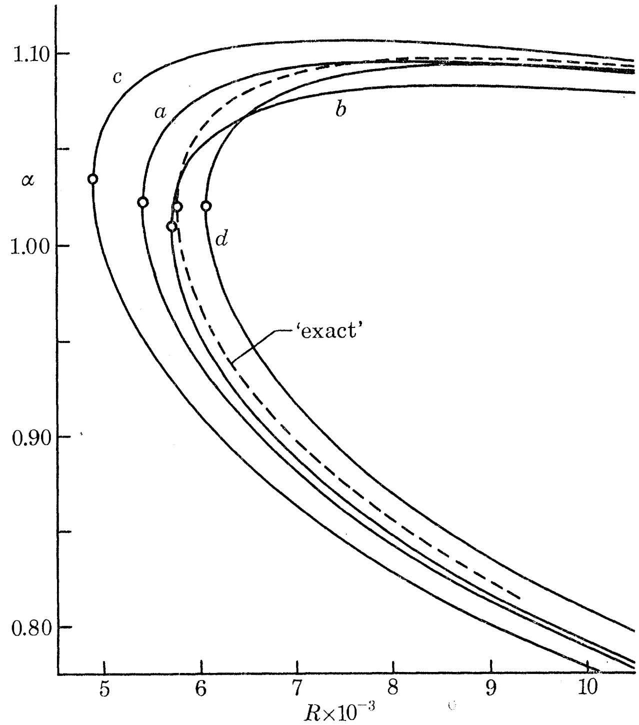

Figure 4. The curves of marginal stability for plane Poiseuille flow according to the heuristic approximations to the eigenvalue relation. The approximations used for $\phi_3(z)$ are (*a*) the local turning point approximation, (*b*) Tollmien's improved viscous approximation, (*c*) uniform approximation (truncated equation) and (*d*) uniform approximation (Orr–Sommerfeld equation). The circled points correspond to the values of α_c and R_c. The dashed curve is based on 'exact' numerical values obtained by Reynolds & Potter (1967), Orszag (1971) and T. H. Hughes (unpublished); on this scale it is indistinguishable from the results obtained by using the fully uniform approximation (7.5).

(*b*) *Tollmien's improved approximation to* $\phi_3(z)$

In the derivation of the inner approximation (8.2) and the first term of the outer approximation (2.10), the only terms in the Orr–Sommerfeld equation which contribute are $(\mathrm{i}\alpha R)^{-1}\phi^{\mathrm{iv}}$ and $(U-c)\,\phi''$. This suggests that, in approximating ϕ_3, it may be permissible to start with the 'truncated' equation

$$(\mathrm{i}\alpha R)^{-1}\phi^{\mathrm{iv}} - (U-c)\,\phi'' = 0. \tag{8.6}$$

This is a second-order equation for ϕ'' to which all of the standard asymptotic theory for second-order equations with a simple turning point is directly applicable, and a casual application of that theory (Shen 1964; Reid 1965) leads to Tollmien's 'improved' approximation

$$\phi_3(z) \sim \eta'^{-\frac{5}{2}} A_1(\zeta, 2). \tag{8.7}$$

In differentiating this approximation, only the term which arises from the differentiation of $A_1(\zeta, 2)$ can consistently be retained and, in this approximation, equation (8.1) becomes

$$\Delta^{(0)}(z) \sim \eta'^{-\frac{5}{2}}\{\eta'\Phi^{(0)}(z) - \eta\Phi^{(0)\prime}(z)\} A_1(\zeta, 1) + \epsilon\eta'^{-\frac{5}{2}}\Phi^{(0)\prime}(z)\, A_1(\zeta, -1). \tag{8.8}$$

For computational purposes this can be written in the form

$$\frac{U_1'}{c}\frac{\Phi^{(0)}(z_1)}{\Phi^{(0)\prime}(z_1)} + \{1+\lambda_2(c)\} F(Z) = 0, \tag{8.9}$$

where

$$1+\lambda_2(c) = -\frac{U_1'}{c}\frac{\eta(z_1)}{\eta'(z_1)} = \frac{3}{2}\frac{U_1'}{c^{\frac{3}{2}}}\int_{z_1}^{z_c}(c-U)^{\frac{1}{2}}\,\mathrm{d}z \tag{8.10}$$

and Z is defined by equation (7.2). Calculations based on this approximation to $\Delta^{(0)}(z_1)$ lead to the results given in table 3. The improvement in the value of R_c, compared to the local turning-point approximation, has led to the widespread belief that (8.7) is a significantly better approximation to ϕ_3 than (8.2). This improvement is somewhat misleading, however, as shown by the results given in figure 4.

To assess the errors associated with Tollmien's approximation to ϕ_3, consider equation (3.16) which, with $k = 1$, can be written in the alternate form

$$\phi_3(z) \sim \eta'^{-\frac{5}{2}} A_1(\zeta, 2) - \epsilon(2c - \eta b)\, A_1(\zeta, 3) - 3\epsilon^2 b A_1(\zeta, 4).$$

This shows that when $\eta \in \mathbf{T}_2 \cup \mathbf{T}_3$, the relative errors associated with the approximation (8.7) are $O(\epsilon, \epsilon^{\frac{6}{5}}, \epsilon^{\frac{3}{2}})$. When $\eta \in \mathbf{T}_1 \cap \mathbf{S}_2$, i.e. in the sector containing η_1, equations (3.16) and (8.7) can each be written as the sum of dominant, balanced and recessive terms. The relative errors associated with the dominant and recessive terms in (8.7) remain unchanged, as would be expected, but those associated with the balanced term are $O(\epsilon, \epsilon^{\frac{3}{5}}, 1)$. Thus, equation (8.7) does not provide a uniform approximation to the solution of either the truncated equation (8.6) or the Orr–Sommerfeld equation (2.1).

(c) *The uniform approximation to $\phi_3(z)$ based on the truncated equation*

To derive uniform approximations to the solutions of the truncated equation (8.6) we again make the preliminary transformation (3.1) with $m = 0$. This leads to an equation of the form (3.3) where the coefficients are now given by

$$\left.\begin{aligned} f_0(\eta) &= 6\gamma, & f_1(\eta) &= -(4\gamma' + 11\gamma^2),\\ g_0(\eta) &= \eta\gamma, & g_1(\eta) &= -(\gamma'' + 7\gamma'\gamma + 6\gamma^3),\\ h_0(0) &= 0, & h_1(\eta) &= 0. \end{aligned}\right\} \tag{8.11}$$

Let $\hat{V}_k(\eta)$ denote the solutions of this equation which are of dominant-recessive type. Then they must also have uniform approximations of the form (3.16) and a short calculation shows that the slowly varying coefficients in the approximations are given by

$$\left.\begin{aligned} a(\eta) &= (z - z_c)/\eta, \quad a(\eta) + \eta c(\eta) = \eta'^{-\frac{5}{2}},\\ b(\eta) &= 2\eta^{-1}c(\eta) + \eta'^{-\frac{5}{2}}\eta^{-\frac{1}{2}}\{G_1(z) - \tfrac{101}{48}\eta^{-\frac{3}{2}}\}, \end{aligned}\right\} \tag{8.12}$$

and

where $G_1(z)$ is now given by [cf. equation (2.13)]

$$G_1(z) = \left(\frac{101}{48}\frac{U'}{U-c} - \frac{1}{24}\frac{U''}{U'}\right)\left(\frac{U'_c}{U-c}\right)^{\frac{1}{2}} + \frac{1}{24}\int_{z_0}^{z}\left(\frac{U'''}{U'} - \frac{U''^2}{U'^2}\right)\left(\frac{U'_c}{U-c}\right)^{\frac{1}{2}} dz. \tag{8.13}$$

If we now approximate $\epsilon\hat{V}_1$ and $\epsilon\hat{V}'_1$ by $\mathscr{L}A_1(\zeta, 1)$ and $\mathscr{M}A_1(\zeta, 1)$ respectively, where a, ℓ and c are given by equations (8.12), then equation (8.1) becomes

$$\begin{aligned}\Delta^{(0)}(z) \sim \{&\Phi^{(0)}(z) - (z-z_c)\,\Phi^{(0)\prime}(z)\}\,A_1(\zeta, 1)\\ &+\epsilon\eta^{-1}\eta'^{-\frac{3}{2}}\{\Phi^{(0)}(z) - \eta'^{\frac{3}{2}}[\Phi^{(0)}(z) - (z-z_c)\,\Phi^{(0)\prime}(z)]\}\,A_1(\zeta, -1)\\ &+\epsilon^2\{\eta'\Phi^{(0)}(z)\,(\ell + c') - \Phi^{(0)\prime}(z)\,c\}\,A_1(\zeta, 0)\}.\end{aligned} \tag{8.14}$$

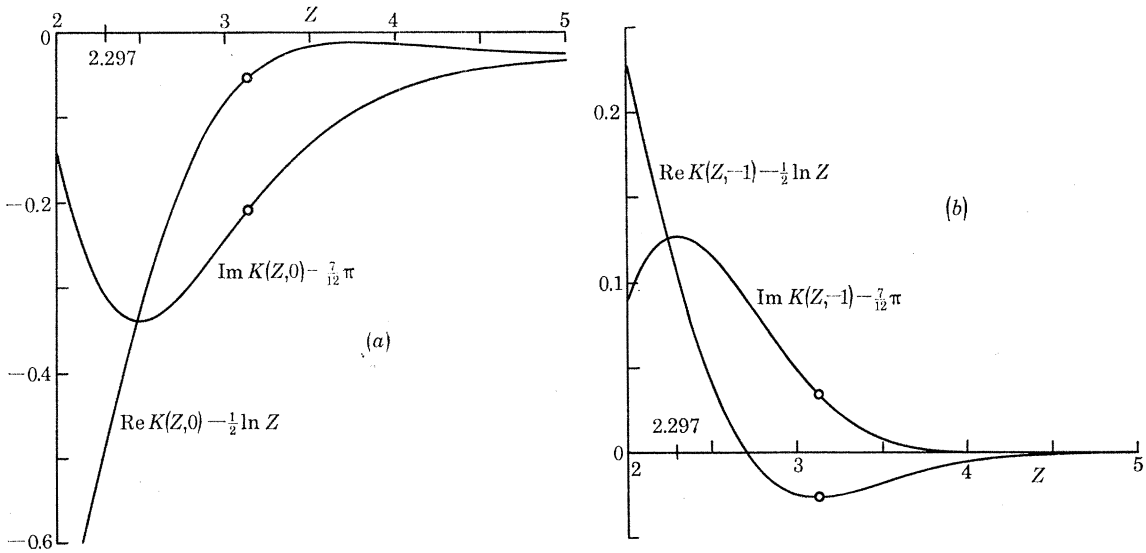

FIGURE 5. The behaviour of the 'viscous corrections' $K(Z,p) - (\frac{1}{2}\ln Z + \frac{7}{12}\pi i)$ for $p = 0$ and -1. The circled points correspond to the minimum critical Reynolds number and $Z = 2.297$ corresponds to the asymptote to the lower branch of the marginal curve.

Although the structure of this approximation is similar to that of equation (5.13) it leads, as table 3 and figure 4 clearly show, to surprisingly poor results. The approximation (8.14) is, of course, defective in two respects since the approximations to ϕ_2 and ϕ_3 used in its derivation are not uniform.

(d) *The uniform approximation to $\phi_3(z)$ based on the Orr–Sommerfeld equation*

To assess the relative effects of these two defects, consider now an approximation to $\Delta^{(0)}(z)$ in which $\epsilon\phi_3$ and $\epsilon\phi'_3$ are approximated by $\mathscr{L}A_1(\zeta, 1)$ and $\eta'\mathscr{M}A_1(\zeta, 1)$ respectively, where a, ℓ and c are given by equations (3.22). This leads immediately to

$$\begin{aligned}\Delta^{(0)}(z) \sim A_1(\zeta, 1) + \epsilon\eta^{-1}\eta'^{-\frac{3}{2}}\{\Phi^{(0)}(z) - \eta'^{\frac{3}{2}}\}\,A_1(\zeta, -1)&\\ +\epsilon^2\{\eta'\Phi^{(0)}(z)\,(\ell + c') - \Phi^{(0)\prime}(z)\,c\}\,A_1(\zeta, 0).&\end{aligned} \tag{8.15}$$

For computational purposes this can be written in the form

$$\begin{aligned}\frac{e^{\frac{5}{6}\pi i}}{A_1(\zeta_1, 0)}\Delta^{(0)}(z_1) \sim H(Z, 1) + \eta'^{-\frac{3}{2}}\{\Phi^{(0)}(z_1) - \eta'^{\frac{3}{2}}\}\,H(Z, -1)&\\ +\epsilon^2 e^{\frac{5}{6}\pi i}\{\eta'\Phi^{(0)}(z_1)\,(\ell + c') - \Phi^{(0)\prime}(z_1)\,c\}.&\end{aligned} \tag{8.16}$$

This approximation to the eigenvalue relation differs from the uniform approximation (7.5) only in the absence of the 'viscous corrections' $K(Z,p)-(\frac{1}{2}\ln Z+\frac{7}{12}\pi i)$ whose behaviour is shown in figure 5 for $p=0$ and -1. Although the differences between the approximations (7.5) and (8.16) might appear to be small, they obviously are of crucial importance numerically as the results given in table 3 and figure 4 clearly show.

Thus, it would appear that equation (7.5) provides a 'first' uniform approximation to the eigenvalue relation for the Orr–Sommerfeld problem and that further simplification is not possible without destroying the uniformity of the approximation.

9. Discussion

Throughout this paper we have, for simplicity, considered only the Orr–Sommerfeld equation but it would appear that the basic ideas involved are applicable, with some generalizations, to a wide variety of problems in hydrodynamic stability. One important example would be the system of equations, of order six, which governs the stability of compressible boundary layers. Heuristic approximations to the eigenvalue relation for this problem are found to fail at high Mach numbers and this failure has usually been attributed (Lees & Reshotko 1962; Shen 1964) to the sharp distinction which is made in the heuristic theory between the rapidly varying 'viscous' solutions and the slowly varying 'inviscid' solutions. No such distinction is made in the present theory since all viscous effects are fully included in the uniform approximations to both the solutions and the eigenvalue relation. Some progress has already been made in this problem by Ng (1976) who derived uniform 'first approximations' to the solutions of the Dunn–Lin equations. Another example would be the equation, also of order six, which governs the stability of stratified viscous shear flows. Some partial results have recently been obtained by Davey & Reid (1977) in the special case of stratified plane Couette flow with a constant buoyancy frequency and a Prandtl number of one. Neither of these problems lends itself to treatment by the comparison equation method, but the derivation of first approximations to the eigenvalue relation, along the lines described in this paper, appears much more promising.

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